Real Description of Classical Hamiltonian Dynamics Generated by a Complex Potential

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Abstract

Analytic continuation of the classical dynamics generated by a standard Hamiltonian, $H = \frac{\mathrm{p}^2}{2m} + v(\mathrm{x})$, into the complex plane yields a particular complex classical dynamical system. For an analytic potential v, we show that the resulting complex system admits a description in terms of the phase space $\mathbb{R}^4$ equipped with an unconventional symplectic structure. This in turn allows for the construction of an equivalent real description that is based on the conventional symplectic structure on $\mathbb{R}^4$, and establishes the equivalence of the complex extension of classical mechanics that is based on the above-mentioned analytic continuation with the conventional classical mechanics. The equivalent real Hamiltonian turns out to be twice the real part of H, while the imaginary part of H plays the role of an independent integral of motion ensuring the integrability of the system. The equivalent real description proposed here is the classical analog of the equivalent Hermitian description of unitary quantum systems defined by complex, typically $\mathcal{PT}$-symmetric, potentials.

Keywords: Hamiltonian dynamics, complex potential, symplectic structure, $\mathcal{PT}$-symmetry

1 Introduction

The recent discovery that the standard quantum Hamiltonian operators,

$$\hat{H} = \frac{\hat{p}^2}{2m} + v(\hat{x}), \tag{1}$$

with certain complex potentials such as $v(x) = ix^3$ have a purely real spectrum [1] has triggered a thorough investigation of quantum systems defined by such Hamiltonians. An important

outcome of this investigation is that at least for the cases that the spectrum is discrete the reality of the spectrum is not only necessary [2] but also sufficient [3] for the existence of a positive-definite inner product that renders the quantum dynamics unitary [4].[1] As shown in [3, 4] under these conditions the Hamiltonian turns out to be quasi-Hermitian [8]. This in turn leads to another crucial finding namely that the resulting unitary quantum system admits an equivalent Hermitian description [9]. The latter can be used to define an underlying classical Hamiltonian system whose pseudo-Hermitian quantization yields the initial quantum system [10, 11, 12]. This construction of an underlying classical system for the quantum Hamiltonian (1) is fundamentally different from a direct association of this Hamiltonian operator with the complex classical Hamiltonian

$$H = \frac{\mathrm{p}^2}{2m} + v(\mathrm{x}). \tag{2}$$

The purpose of this article is to investigate the possibility of a real description of the complex dynamical systems generated by the Hamiltonians of the form (2) where $v : \mathbb{C} \to \mathbb{C}$ is an analytic function.

Specific examples of these complex classical systems have been studied in [13, 14, 15, 16], and a complex phase space approach has been proposed in [17]. But, to the best of our knowledge, a thorough investigation of the associated symplectic structure(s) and the relation to the conventional real classical dynamical systems has not been reported previously.

For a complex-valued potential the Hamiltonian dynamics defined by (2) takes place in a complex phase space. Hence we shall use $\mathfrak{z}$ and $\mathfrak{p}$ to denote the complex dynamical phase-space coordinates x and p, respectively. In this notation, the classical Hamiltonian reads

$$H = \frac{\mathfrak{p}^2}{2m} + v(\mathfrak{z}), \qquad \mathfrak{z}, \mathfrak{p} \in \mathbb{C}. \tag{3}$$

We will also introduce

$$x := \Re(\mathfrak{z}), \qquad y := \Im(\mathfrak{z}), \qquad p := \Re(\mathfrak{p}), \qquad q := \Im(\mathfrak{p}), \tag{4}$$

$$v_r(x,y) := \Re(v(x+iy)), \qquad v_i(x,y) := \Im(v(x+iy)), \tag{5}$$

$$H_r := \Re(H) = \frac{p^2 - q^2}{2m} + v_r(x,y), \qquad H_i := \Im(H) = \frac{pq}{m} + v_i(x,y), \tag{6}$$

where $\Re$ and $\Im$ stand for the real and imaginary parts of their argument, respectively. Note also that because v is assumed to be a (complex) analytic function, v_r and v_i satisfy the Cauchy-Riemann conditions:

$$\partial_x v_r(x,y) = \partial_y v_i(x,y), \qquad \partial_y v_r(x,y) = -\partial_x v_i(x,y). \tag{7}$$

[1] This inner product is not unique [5, 6]. A particular example is the $\mathcal{CPT}$-inner product proposed in [7].

2 Compatible Symplectic Structures

The complex Hamiltonian (3) defines a dynamics in the complex phase space $\mathbb{C}^2$ according to the Hamilton's equations

$$\dot{\mathfrak{z}} = \partial_{\mathfrak{p}} H = \frac{\mathfrak{p}}{m}, \qquad \dot{\mathfrak{p}} = -\partial_{\mathfrak{z}} H = -\partial_{\mathfrak{z}} v(\mathfrak{z}), \tag{8}$$

where a dot denotes a time-derivative, and the time parameter t is assumed to take real values. Our aim in this section is to determine the symplectic structures [18] on the phase space $\mathfrak{P} = \mathbb{C}^2$ that are compatible with the dynamical equations (8). In other words, we wish to construct a Poisson-like bracket (an antisymmetric, non-degenerate, bilinear form also called skew inner product [18]) $\{\!\{\cdot,\cdot\}\!\}$ in terms of which (8) takes the form

$$\dot{\mathfrak{z}} = \{\!\{\mathfrak{z}, H\}\!\}, \qquad \dot{\mathfrak{p}} = \{\!\{\mathfrak{p}, H\}\!\}. \tag{9}$$

Before addressing this problem, however, we shall first show that the choice of the standard symplectic structure, i.e., setting $\mathfrak{P} = \mathbb{R}^4$ and endowing it with the standard symplectic structure, is not consistent with the dynamical equations (8).

The standard symplectic structure on $\mathbb{R}^4 = \mathbb{C}^2$ is defined by the conventional Poisson bracket $\{\cdot,\cdot\}$ according to

$$\begin{aligned} \{A, B\} &:= (\partial_x A\, \partial_p B + \partial_y A\, \partial_q B) - (A \leftrightarrow B) \\ &= 2(\partial_{\mathfrak{z}} A\, \partial_{\mathfrak{p}^*} B + \partial_{\mathfrak{z}^*} A\, \partial_{\mathfrak{p}} B) - (A \leftrightarrow B), \end{aligned} \tag{10}$$

where $A, B : \mathfrak{P} \to \mathbb{C}$ are smooth functions, $(A \leftrightarrow B)$ stands for the preceding terms with A and B exchanged, and we have made use of the identities

$$\partial_{\mathfrak{z}} = \frac{1}{2}(\partial_x - i\partial_y), \qquad \partial_{\mathfrak{z}^*} = \frac{1}{2}(\partial_x + i\partial_y). \tag{11}$$

Clearly, in view of (3) and (10),

$$\dot{\mathfrak{z}} = \{\mathfrak{z}, H\} = 0, \qquad \dot{\mathfrak{p}} = \{\mathfrak{p}, H\} = 0.$$

Hence, a symplectic structure that is consistent with the dynamical equations (8), if exists, is not the standard one. It is this observation that motivates the search for finding dynamically compatible nonstandard symplectic structures. To the best of our knowledge the first step in this direction is taken in [15] where the authors briefly discuss the issue and give a special class of compatible symplectic structures. In the following we offer a thorough and systematic investigation of the compatible symplectic structures.

To construct a compatible symplectic structure on the phase space we recall using the defining properties of $\{\!\{\cdot,\cdot\}\!\}$ that

$$\{\!\{A, B\}\!\} = \sum_{i,j=1}^{4} \mathcal{J}_{ij}\, \partial_{\mathfrak{w}_i} A\, \partial_{\mathfrak{w}_j} B, \tag{12}$$

where $\mathfrak{w}_1 := \mathfrak{z}, \mathfrak{w}_2 := \mathfrak{p}, \mathfrak{w}_3 := \mathfrak{z}^*, \mathfrak{w}_4 := \mathfrak{p}^*$, and $\mathcal{J}_{ij}$ are components of a symplectic form $\omega_{\mathcal{J}}$ or the entries of the associated invertible antisymmetric matrix $\mathcal{J}$. The latter is sometimes called a symplectic matrix [19].

Imposing the reality condition,

$$\{\!\{A, B\}\!\}^* = \{\!\{A^*, B^*\}\!\}, \tag{13}$$

and requiring the compatibility with the dynamical equations (8) and non-degeneracy of $\omega_{\mathcal{J}}$ (equivalently invertibility of $\mathcal{J}$), we find

$$\{\!\{\mathfrak{z}, \mathfrak{p}\}\!\} = 1, \qquad \{\!\{\mathfrak{z}, \mathfrak{z}^*\}\!\} = ia, \qquad \{\!\{\mathfrak{z}, \mathfrak{p}^*\}\!\} = \alpha, \tag{14}$$

$$\{\!\{\mathfrak{p}, \mathfrak{z}^*\}\!\} = -\alpha^*, \qquad \{\!\{\mathfrak{p}, \mathfrak{p}^*\}\!\} = ib, \qquad \{\!\{\mathfrak{z}^*, \mathfrak{p}^*\}\!\} = 1, \tag{15}$$

where $a, b \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ such that $|\alpha|^2 - ab \neq 1$. Eqs. (14) and (15) together with the antisymmetry of $\{\!\{\cdot, \cdot\}\!\}$ determines the latter in terms of the free parameters a, b, α. Specifically, $\{\!\{\cdot, \cdot\}\!\}$ satisfies (12) with $\mathcal{J}$ given by

$$\mathcal{J} = \begin{pmatrix} 0 & 1 & ia & \alpha \\ -1 & 0 & -\alpha^* & ib \\ -ia & \alpha^* & 0 & 1 \\ -\alpha & -ib & -1 & 0 \end{pmatrix}. \tag{16}$$

In order to see if $\{\!\{\cdot, \cdot\}\!\}$ defines a real symplectic structure on $\mathbb{R}^4$, we introduce

$$w_1 := x, \qquad w_2 := p, \qquad w_3 := y, \qquad w_4 = q, \tag{17}$$

and express $\{\!\{\cdot, \cdot\}\!\}$ as

$$\{\!\{A, B\}\!\} = \sum_{i,j=1}^{4} J_{ij}\ \partial_{w_j} A\, \partial_{w_j} B, \tag{18}$$

where J_{ij} depend on $\mathcal{J}_{ij}$. A straightforward calculation using (11), (12), (16), and (18) identifies J_{ij} with the entries of

$$J = \frac{1}{2} \begin{pmatrix} 0 & 1+\alpha_r & -a & -\alpha_i \\ -(1+\alpha_r) & 0 & -\alpha_i & -b \\ a & \alpha_i & 0 & -1+\alpha_r \\ \alpha_i & b & 1-\alpha_r & 0 \end{pmatrix}, \tag{19}$$

where $\alpha_r := \Re(\alpha)$ and $\alpha_i := \Im(\alpha)$. As seen from (19), J is a real invertible antisymmetric (symplectic) matrix, and $\{\!\{\cdot, \cdot\}\!\}$ defines a genuine symplectic structure on $\mathbb{R}^4$ that is by construction compatible with the dynamical equations (8). It is not difficult to see that indeed (19) is the most general symplectic matrix with these properties.

The standard symplectic structure, that is defined using the symplectic matrix

$$J_{\rm st} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{20}$$

does not fulfil (19). The simplest example of the allowed symplectic matrices (19) is

$$J_0 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{21}$$

which corresponds to the choice $a = b = \alpha = 0$.

3 Equivalent Formulation Using Standard Symplectic Structure

The fact that $J_{\rm st}$ fails to belong to the class of symplectic matrices (19) does not mean that the latter are associated with fundamentally different theories. According to the well-known uniqueness theorem for the symplectic structures on $\mathbb{R}^{2n}$, every symplectic structure is isomorphic to the standard one [18].

For the case at hand, it is not difficult to find a similarity transformation $J \to J' = S^{-1}JS$, by a real orthogonal matrix S, that maps J to[2]

$$J' := \begin{pmatrix} 0 & r_+ & 0 & 0 \\ -r_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & r_- \\ 0 & 0 & -r_- & 0 \end{pmatrix}, \tag{22}$$

where

$$r_\pm := \sqrt{\frac{1}{8}\left(a^2 + b^2 + 2(|\alpha|^2+1) \pm \sqrt{[(a+b)^2+4][(a-b)^2+4|\alpha|^2]}\right)} \in \mathbb{R}^+.$$

Hence the following new coordinates in $\mathbb{R}^4$ serve as the symplectic (Darboux) coordinates associated with the symplectic matrix J.

$$x_1 = r_+^{-1/2}\sum_{k=1}^{4} S_{k1}w_k, \quad p_1 = r_+^{-1/2}\sum_{k=1}^{4} S_{k2}w_k, \quad x_2 = r_-^{-1/2}\sum_{k=1}^{4} S_{k3}w_k, \quad p_2 = r_-^{-1/2}\sum_{k=1}^{4} S_{k4}w_k, \tag{23}$$

[2] J has four linearly independent eigenvectors $\vec{v}_+, \vec{v}_+^*, \vec{v}_-, \vec{v}_-^*$. The columns of S are unit vectors aligned along the real and imaginary parts of $\vec{v}_\pm$. S is orthogonal, because the eigenvectors of J are orthogonal.

where we used the fact that S is orthogonal.

As explicit expressions for the symplectic coordinates (23) are complicated, we will here suffice to present them only for the simplest case, namely $a = b = \alpha = 0$, that corresponds to the symplectic matrix J_0. In this case, we have

$$x_1 = \sqrt{2}\,w_1 = \sqrt{2}\,x, \quad p_1 = \sqrt{2}\,w_2 = \sqrt{2}\,p, \quad x_2 = \sqrt{2}\,w_4 = \sqrt{2}\,q, \quad p_2 = \sqrt{2}\,w_3 = \sqrt{2}\,y, \tag{24}$$

which are essentially identical with those initially considered in [20]. See also [17].[3]

Having obtained a set of symplectic coordinates associated with a dynamically compatible symplectic structure, we can express the dynamical equations (8) in terms of a set of standard Hamilton equations, namely

$$\dot{x}_1 \;=\; \{x_1, h\} = \frac{p_1}{m}, \qquad \dot{x}_2 = \{x_2, h\} = 2\;\partial_{p_2} v_r(2^{-\frac{1}{2}}x_1, 2^{-\frac{1}{2}}p_2), \tag{25}$$

$$\dot{p}_1 \;=\; \{p_1, h\} = -2\;\partial_{x_1} v_r(2^{-\frac{1}{2}}x_1, 2^{-\frac{1}{2}}p_2), \qquad \dot{p}_2 = \{p_2, h\} = \frac{x_2}{m}, \tag{26}$$

for the real Hamiltonian

$$h := \frac{p_1^2 - x_2^2}{2m} + 2\,v_r(2^{-\frac{1}{2}}x_1, 2^{-\frac{1}{2}}p_2) = 2H_r. \tag{27}$$

One can check using the (3) – (7) and (11) that (25) – (26) are equivalent to (8).[4] In particular, the structure of the trajectories in the x-y (equivalently x_1-p_2) plane for the $\mathcal{PT}$-symmetric potentials $v(\mathfrak{z}) = -(i\mathfrak{z})^n$ (with $n \in \mathbb{Z}$)[5] and $v(\mathfrak{z}) = \sum_{k>0} \mu_k e^{ik\mathfrak{z}}$ (with $\mu_k \in \mathbb{R}$) that are respectively examined in [13, 16] and [15] can be obtained using the real Hamiltonian (27).

As expected H_r which is half the Hamiltonian h is an integral of motion. The same is true about

$$H_i = \frac{x_2 p_1}{2m} + v_i(2^{-\frac{1}{2}}x_1, 2^{-\frac{1}{2}}p_2), \tag{28}$$

i.e., $\dot{H}_i = \{H_i, h\} = 0$.[6] It provides an independent integral of motion for the system that ensures its integrability via Liouville's theorem [18]. What has been done in the recent studies of $\mathcal{PT}$-symmetric potentials [13, 14, 16] is to set the value of H_i to zero and study the behavior of the solutions satisfying this constraint. Table 1 gives the explicit form of the real Hamiltonian h and the invariant H_i for some typical $\mathcal{PT}$-symmetric potentials.

The invariant H_i generates a set of symmetry transformations in the phase space. The

[3]Note that unlike in [20, 17] where these coordinates were introduced essentially for convenience, we offer a systematic derivation of them based on the uniqueness theorem for symplectic structures.

[4]After the completion of this project it was brought to our attention that the observation that the real part of a complex analytic Hamiltonian can generate the dynamics in the coordinates (24) was previously made in [20].

[5]For non-integer n this potential is not an entire function and special care needs be taken whenever a trajectory crosses a branch cut.

[6]It is a straightforward exercise to show using (25), (26), and (28) that $\dot{H}_i = 0$.

$v(\mathfrak{z})$	$h(x_1,x_2,p_1,p_2)$	$H_i(x_1,x_2,p_1,p_2)$
$i\,\mathfrak{z}$	$p_1^2 - p_2/\sqrt{2} - x_2^2$	$x_2p_1 + x_1/\sqrt{2}$
$\mathfrak{z}^2$	$p_1^2 - p_2^2 + x_1^2 - x_2^2$	$x_2p_1 + x_1p_2$
$i\mathfrak{z}^3$	$p_1^2 + (p_2^3 - 3x_1^2p_2)/\sqrt{2} - x_2^2$	$x_2p_1 + (x_1^3 - 3x_1p_2^2)/2\sqrt{2}$
$-\mathfrak{z}^4$	$p_1^2 - (x_1^4 - 6x_1^2p_2^2 + p_2^4)/2 - x_2^2$	$x_2p_1 - x_1^3p_2 - x_1p_2^3$
$e^{i\mathfrak{z}}$	$p_1^2 + 2\,e^{-p_2/\sqrt{2}}\cos(x_1/\sqrt{2}) - x_2^2$	$x_2p_1 + e^{-p_2/\sqrt{2}}\sin(x_1/\sqrt{2})$
$i\sin\mathfrak{z}$	$p_1^2 - 2\cos(x_1/\sqrt{2})\sinh(p_2/\sqrt{2}) - x_2^2$	$x_2p_1 + \sin(x_1/\sqrt{2})\cosh(p_2/\sqrt{2})$

Table 1: Equivalent real Hamiltonian h and the integral of motion H_i for various $\mathcal{PT}$-symmetric analytic potentials v. m is set to 1/2.

infinitesimal symmetry transformations have the form

$$x_1 \;\to\; x_1 + \epsilon\{x_1, H_2\} = x_1 + \epsilon(\frac{x_2}{2m}), \tag{29}$$

$$x_2 \;\to\; x_2 + \epsilon\{x_2, H_2\} = x_2 + \epsilon\ \partial_{x_1} v_r(2^{-\frac{1}{2}}x_1, 2^{-\frac{1}{2}}p_2), \tag{30}$$

$$p_1 \;\to\; p_1 + \epsilon\{p_1, H_2\} = p_1 + \epsilon\ \partial_{p_2} v_r(2^{-\frac{1}{2}}x_1, 2^{-\frac{1}{2}}p_2), \tag{31}$$

$$p_2 \;\to\; p_2 + \epsilon\{p_2, H_2\} = p_2 - \epsilon(\frac{p_1}{2m}), \tag{32}$$

where ϵ is an infinitesimal real variable.

4 Summary and Conclusions

Analytic continuation of a potential defined on the real axis to complex plane determines a complex Hamiltonian dynamical system having two real configurational degrees of freedom and four phase space degrees of freedom. The condition that the symplectic structure on the phase space $\mathbb{C}^2 = \mathbb{R}^4$ be compatible with the dynamical equations restricts the former to a four-parameter family of symplectic structures which does not include the standard symplectic structure. Nevertheless, all these structures are isomorphic to the standard symplectic structure. This implies the existence of a conventional description of the complex systems using a real Hamiltonian that turns out to be twice the real part of initial complex Hamiltonian H. The imaginary part of H is an integral of motion rendering the system integrable.

In the study of $\mathcal{PT}$-symmetric potentials, the imaginary part of the classical Hamiltonian is often set to zero. This yields certain special classical trajectories whose physical superiority over those having $H_i \neq 0$ is not clear. The situation resembles confining the study of the trajectories of coulomb potential to those having a particular value of angular momentum and ignoring the others.

It is important to note that the classical dynamics determined by the analytic continuation of Hamilton's equations defines a classical system whose standard canonical quantization is different from the one corresponding to the naive prescription

$$\mathfrak{z} \to \hat{x}, \qquad \mathfrak{p} \to \hat{p}, \qquad \{\cdot,\cdot\} \to -i[\cdot,\cdot], \tag{33}$$

where $\hat{x}$ and $\hat{p}$ are the usual position and momentum operators, $\{\cdot,\cdot\}$ is the Poisson bracket, and $[\cdot,\cdot]$ is the commutator. One of the reasons for this is that the symplectic structure associated with the Poisson bracket is not compatible with the classical dynamical equations. As a result the Heisenberg equations do not tend to the Hamilton equations involving the usual Poisson bracket in the classical limit. Another reason is that the complex classical system is intrinsically two-dimensional (having a four-dimensional phase space) whereas the quantum system with the Hamiltonian $\hat{H} = \hat{p}^2/2m + v(\hat{x})$ is one-dimensional. One can insist on defining an effective one-dimensional system by enforcing $H_i = 0$ as a constraint and moding out the symmetry transformations (29) – (32) it generates to construct a two-dimensional reduced phase space [19, 21]. Whether this reduced system is related to the one corresponding to the classical limit of the equivalent Hermitian Hamiltonian operator [9, 10, 11] is an interesting question worthy of investigation. The relation, if there is one, is expected not to be direct, for we know that for complex analytic potentials with a non-real spectrum there is no equivalent Hermitian Hamiltonian operator, whereas the classical equivalent real Hamiltonian can always be constructed.

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